

REPRESENTATION OF QUANTUM STATES AS POINTS IN A PROBABILITY SIMPLEX ASSOCIATED TO A SIC-POVM

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ABSTRACT. The quantum state of a d -dimensional system can be represented by a probability distribution over the d^2 outcomes of a Symmetric Informationally Complete Positive Operator Valued Measure (SIC-POVM), and then this probability distribution can be represented by a vector of \mathbb{R}^{d^2-1} in a $(d^2 - 1)$ -dimensional simplex, we will call this set of vectors \mathcal{Q} . Other way of represent a d -dimensional system is by the corresponding Bloch vector also in \mathbb{R}^{d^2-1} , we will call this set of vectors \mathcal{B} . In this paper it is proved that with the adequate scaling $\mathcal{B} = \mathcal{Q}$. Also we indicate some features of the shape of \mathcal{Q} .

1. INTRODUCTION

In quantum mechanics a quantum state is described by a density operator, ρ , but there are alternative descriptions. One alternative description is to parameterize the space of density matrices with Bloch vectors and study the structure formed by these vectors, \mathcal{B} , this is done thoroughly in [6] and [7]. Other possible description is provided by the probabilities $p_i = \text{Tr}(E_i \rho)$, where E_i are the elements of a SIC-POVM [8], an informationally complete and symmetric POVM, with the minimal number of elements. This is the description of quantum states chosen in Quantum-Bayesianism [3].

According to the quantum-Bayesian approach to quantum foundations, see for example [4], probabilities p_i represent an agents Bayesian degrees of belief. When we represent the probabilities of a SIC-POVM as points in the corresponding probability simplex, Δ , we will see that these probabilities are not arbitrary, not any point of Δ can represent a quantum state, only a proper subset $\mathcal{Q} \subset \Delta$. The problem is then to understand the structure of \mathcal{Q} , if it is possible, in physical terms. Why are the beliefs of our agent constrained in this way? I think a hint to the answer to this question is to realize that the structure of \mathcal{Q} is agent independent, of course for a particular experiment two agents can differ in the assignments of probabilities for the outcomes of a SIC-POVM, but the two distributions will be represented by points in \mathcal{Q} . Then the structure of \mathcal{Q} is saying us something about the external world, at least about the intersubjective world.

The main result of this paper is demonstrate that $\mathcal{B} = \mathcal{Q}$, with suitable scalings. We therefore can translate the results obtained in the study of \mathcal{B} to probability concepts, and I think this is a useful translation because in terms of probabilities we can made use of the tools of Quantum Information Theory and our physical intuition in trying to understand the *why* of the \mathcal{Q} structure.

The structure of this paper is as follows. In SECTION 2 we review some basic facts about simplexes as geometrical objects that can represent probability distributions. In SECTION 3 we review the Bloch representation of quantum states and the conditions these vectors satisfy. In SECTION 4 we give the Bloch representation

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of a SIC-POVM, and see what conditions satisfy the corresponding Bloch vectors. In SECTION 5 we construct \mathcal{Q} and demonstrate its equality with \mathcal{B} , we also give some features of the shape of this set.

2. SIMPLEXES AND PROBABILITY DISTRIBUTIONS

This section contains known facts about simplexes and probability distributions, but serves to collect useful results and to fix some conventions in this paper.

Definition 1. *An n -dimensional and regular simplex in \mathbb{R}^n is any set, Δ_{n+1} , that can be defined as follows*

$$\Delta_{n+1} = \left\{ \mathbf{s} = \sum_{i=1}^{n+1} p_i \mathbf{t}_i : p_i \in [0, 1], \sum_{i=1}^{n+1} p_i = 1, \mathbf{t}_i \in \mathbb{R}^n \text{ and} \right.$$

$$\left. \mathbf{t}_i \cdot \mathbf{t}_j = (a - b)\delta_{ij} + b, \ i, j \in \{1, 2, \dots, n+1\} \text{ and } b \neq a \neq 0 \right\}.$$

That is, Δ_{n+1} is the convex hull of the set of vectors $V = \{\mathbf{t}_i\}_{i=1}^{n+1}$, the position vectors of the vertices of Δ_{n+1} .

Some observations about this definition:

First: Definition 1 provides a bijective map from probability distributions over $n+1$ outcomes, $\{p_i\}_{i=1}^{n+1}$, to Δ_{n+1} .

Second: $\mathbf{t}_i^2 = a$, all vertices are at equal distance from the origin, a necessary condition to obtain a regular simplex centered at the origin, and this distance is not zero so that the simplex is not a point.

Third: $\mathbf{t}_i \cdot \mathbf{t}_j = b$ when $i \neq j$, this implies that the vectors of V spread uniformly in space. The condition $b \neq a$ is necessary so that the vectors in V are different. Also we can deduce that $b \neq 0$ because we cannot have $n+1$ orthogonal and no null vectors in \mathbb{R}^n .

Now we will deduce some important properties.

Proposition 1. *Every subset of V that contains n vectors is a basis of \mathbb{R}^n .*

Proof. Given the symmetry between the vectors of V we can, without loss of generality, prove the proposition for the subset $\{\mathbf{t}_i\}_{i=1}^n$. Therefore we have to prove that

$$\sum_{i=1}^n \lambda_i \mathbf{t}_i = \mathbf{0}, \tag{1}$$

has only the trivial solution. We multiply (1) by \mathbf{t}_{n+1} and the result is

$$b \sum_{i=1}^n \lambda_i = 0,$$

and because $b \neq 0$

$$\sum_{i=1}^n \lambda_i = 0. \tag{2}$$

Now we multiply (1) by \mathbf{t}_j , where $j \in \{1, 2, \dots, n\}$

$$\begin{aligned} \sum_{i=1}^n \lambda_i \mathbf{t}_i \cdot \mathbf{t}_j &= 0 \\ \sum_{i=1}^n \lambda_i [(a-b)\delta_{ij} + b] &= 0 \\ (a-b)\lambda_j + b \sum_{i=1}^n \lambda_i &= 0 \\ (a-b)\lambda_j &= 0 && \text{(by equation (2))} \\ \lambda_j &= 0, && \text{(because } a \neq b) \end{aligned}$$

□

Proposition 2. *The sum of all the elements of V is null.*

Proof. By **Proposition 1** we know that $S = \{\mathbf{t}_i\}_{i=1}^n$ is a basis of \mathbb{R}^n , so we can express \mathbf{t}_{n+1} as a linear combination of S-vectors

$$\mathbf{t}_{n+1} = \sum_{i=1}^n \lambda_i \mathbf{t}_i \quad (3)$$

Multiplying (3) by \mathbf{t}_{n+1} we obtain

$$a = b \sum_{i=1}^n \lambda_i. \quad (4)$$

Now we multiply (3) by \mathbf{t}_j , with $j \in \{1, 2, \dots, n\}$

$$\begin{aligned} b &= \sum_{i=1}^n \lambda_i [(a-b)\delta_{ij} + b] \\ b &= (a-b)\lambda_j + a && \text{(by equation (4))}, \end{aligned}$$

and now is immediate that

$$\lambda_j = -1, \quad j \in \{1, 2, \dots, n\}.$$

Substituting in (3)

$$\mathbf{t}_{n+1} = - \sum_{i=1}^n \mathbf{t}_i$$

or

$$\sum_{i=1}^{n+1} \mathbf{t}_i = \mathbf{0}.$$

□

Corollary 1. *The relation between a and b is*

$$b = -\frac{a}{n}. \quad (5)$$

Proof. If in equation (4) we substitute the values found for λ_i we immediately obtain (5). □

Then a remains as a free parameter that fixes the scale of Δ_{n+1} . In view of the last corollary I think that a convenient value for a is $a = n \Rightarrow b = -1$ because then many $1/n$ factors disappear. With these values the inner product between vectors of V reads as follows

$$\mathbf{t}_i \cdot \mathbf{t}_j = (n+1)\delta_{ij} - 1 \quad i, j \in \{1, 2, \dots, n+1\}, \quad (6)$$

that is, $\|\mathbf{t}_i\| = \sqrt{n}$ and $\mathbf{t}_i \cdot \mathbf{t}_j = -1$ when $i \neq j$.

In a regular simplex all the m -facets, m -dimensional facets, are at the same distance from the origin.

Proposition 3. *The distance from the origin to the m -facets is*

$$d_m = \sqrt{\frac{n-m}{m+1}} \quad (7)$$

Proof. Because all m -facets are equidistant from the origin we can take one particular m -facet, for example that defined by the vectors $\{\mathbf{t}_i\}_{i=1}^{m+1}$. The centroid of this m facet is in the position

$$\mathbf{F}_m = \frac{1}{m+1} \sum_{i=1}^{m+1} \mathbf{t}_i$$

then

$$\begin{aligned} d_m^2 &= \mathbf{F}_m^2 = \frac{1}{(m+1)^2} \sum_{i,j=1}^{m+1} \mathbf{t}_i \cdot \mathbf{t}_j \\ &= \frac{1}{(m+1)^2} \sum_{i,j=1}^{m+1} [(n+1)\delta_{ij} - 1] \quad (\text{by equation (6)}) \\ &= \frac{n-m}{m+1} \end{aligned}$$

□

Among these distances are of special interest the radius of the inner sphere and the radius of the outer sphere. The outer sphere is the $(n-1)$ -sphere centered at the origin that contains Δ_{n+1} and such that its radius is minimal, evidently its radius is the distance from the origin to the 0-facets (the vertices of the simplex)

$$R_{out} = d_0 = \|\mathbf{t}_i\| = \sqrt{n}. \quad (8)$$

The inner sphere is the $(n-1)$ -sphere contained in Δ_{n+1} centered at the origin and such that its radius is maximal. From (7) we see that

$$0 = d_n < d_{n-1} < \dots < d_0 = \sqrt{n}, \quad (9)$$

then the inner sphere has radius

$$R_{in} = d_{n-1} = \frac{1}{\sqrt{n}} \quad (10)$$

with a greater radius the sphere would have points situated beyond the $(n-1)$ -facets, and therefore outside Δ_{n+1} .

As we noted **Definition 1** provides a bijective map from probability distributions over $n+1$ outcomes, $\{p_i\}_{i=1}^{n+1}$, to Δ_{n+1} , the map is

$$\begin{aligned} f_{\Delta_{n+1}} : \mathcal{P}_{n+1} &\rightarrow \Delta_{n+1} \\ \{p_i\}_{i=1}^{n+1} &\mapsto \mathbf{s} = \sum_{i=1}^{n+1} p_i \mathbf{t}_i, \end{aligned}$$

where we denote this map by $f_{\Delta_{n+1}}$ because the vectors \mathbf{s} depends on the election of simplex Δ_{n+1} , we have also introduced the symbol \mathcal{P}_{n+1} to denote the set of all probability distributions over $n+1$ outcomes.

In the next proposition we will see how to recover a probability distribution from a given vector $\mathbf{s} \in \Delta_{n+1}$.

Proposition 4. *The inverse of the map defined above is*

$$f_{\Delta_{n+1}}^{-1} : \Delta_{n+1} \rightarrow \mathcal{P}_{n+1}$$

$$\mathbf{s} \mapsto \left\{ p_i = \frac{1}{n+1} (\mathbf{s} \cdot \mathbf{t}_i + 1) \right\}_{i=1}^{n+1} \quad (11)$$

Proof.

$$\mathbf{s} = \sum_{j=1}^{n+1} p_j \mathbf{t}_j$$

$$\mathbf{s} \cdot \mathbf{t}_i = \sum_{j=1}^{n+1} p_j \mathbf{t}_j \cdot \mathbf{t}_i$$

$$\mathbf{s} \cdot \mathbf{t}_i = \sum_{j=1}^{n+1} p_j [(n+1)\delta_{ij} - 1] \quad (\text{by (6)}),$$

Then

$$p_i = \frac{1}{n+1} (\mathbf{s} \cdot \mathbf{t}_i + 1) \quad (12)$$

□

It is also interesting the following relation

Proposition 5. *If $\mathbf{s} \in \Delta_{n+1}$ and $\{p_i\}_{i=1}^{n+1}$ is its corresponding probability distribution, then*

$$\sum_{i=1}^{n+1} p_i^2 = \frac{1}{n+1} (\mathbf{s}^2 + 1) \quad (13)$$

Proof.

$$\mathbf{s}^2 = \sum_{i,j=1}^{n+1} p_i p_j \mathbf{t}_i \cdot \mathbf{t}_j$$

$$= \sum_{i,j=1}^{n+1} p_i p_j [(n+1)\delta_{ij} - 1] \quad (\text{by (6)})$$

$$= (n+1) \sum_{i=1}^{n+1} p_i^2 - 1$$

From which it follows the proposition. □

Corollary 2. *When $\mathbf{s}^2 = d_m^2$ then*

$$\sum_{i=1}^{n+1} p_i^2 = \frac{1}{m+1} \quad (14)$$

3. BLOCH REPRESENTATION OF QUANTUM STATES

We will denote by \mathcal{D}_d the set of density matrices of order d , namely

$$\mathcal{D}_d = \{\rho \in \mathcal{M}_d(\mathbb{C}) : \rho = \rho^\dagger, \rho \geq 0 \text{ and } \text{Tr} \rho = 1\}. \quad (15)$$

Any d -dimensional density matrix ρ can be represented as [5]

$$\rho = \frac{1}{d} + \sqrt{\frac{d+1}{2d}} \mathbf{r} \cdot \boldsymbol{\sigma}, \quad (16)$$

where the coefficient $\sqrt{\frac{d+1}{2d}}$ has been chosen for later convenience. If the vector, $\mathbf{r} \in \mathbb{R}^{d^2-1}$, in (16) is such that the corresponding ρ is a true density matrix, then this vector will be called the Bloch vector associated to ρ . The set of all Bloch vectors in \mathbb{R}^{d^2-1} will be denoted by \mathcal{B}_{d^2-1} . The components of the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_{d^2-1})$ are hermitian matrices with null trace which form a basis of the algebra $\mathfrak{su}(d)$ and we follow the convention that *scalar + square matrix* is read as *scalar* I + *square matrix*, where I is the identity matrix of the appropriate order.

Some useful formulae involving the σ_a 's are, see for example appendix 2 of [2],

$$[\sigma_a, \sigma_b] = 2i f_{abc} \sigma_c, \quad (17)$$

$$\{\sigma_a, \sigma_b\} = \frac{4}{d} \delta_{ab} + 2 d_{abc} \sigma_c, \quad (18)$$

$$\text{Tr}(\sigma_a \sigma_b) = 2 \delta_{ab}, \quad (19)$$

$$\text{Tr}(\sigma_a \sigma_b \sigma_c) = 2 d_{abc} + 2i f_{abc}, \quad (20)$$

where we sum over repeated indices, $f_{abc} \in \mathbb{R}$ is totally antisymmetric, $d_{abc} \in \mathbb{R}$ is totally symmetric, traceless and is identically null when $d = 2$.

In the next proposition we see what conditions have to be fulfilled by \mathbf{r} so that ρ is a pure state. This proposition is enunciated in [2] (eq. 8.24, p. 215) although with a different normalization for the Bloch vectors.

Proposition 6. *ρ is a pure state if and only if the associated Bloch vector, \mathbf{r} , satisfies*

$$\mathbf{r}^2 = \frac{d-1}{d+1} \quad (21)$$

and

$$\mathbf{r} * \mathbf{r} = (d-2) \sqrt{\frac{2}{d(d+1)}} \mathbf{r}, \quad (22)$$

where $(\mathbf{r} * \mathbf{r})_c = d_{abc} r_a r_b$

Proof. ρ is a pure state if and only if $\rho^2 = \rho$

$$\begin{aligned} \rho^2 &= \frac{1}{d^2} + \frac{d+1}{2d} r_a r_b \sigma_a \sigma_b + \frac{1}{d} \sqrt{\frac{2(d+1)}{d}} \mathbf{r} \cdot \boldsymbol{\sigma} \\ &= \frac{1}{d^2} + \frac{d+1}{4d} r_a r_b \{\sigma_a, \sigma_b\} + \frac{1}{d} \sqrt{\frac{2(d+1)}{d}} \mathbf{r} \cdot \boldsymbol{\sigma} \quad (r_a r_b \text{ is symmetric in } a, b) \\ &= \frac{1}{d^2} + \frac{d+1}{4d} r_a r_b \left[\frac{4}{d} \delta_{ab} + 2 d_{abc} \sigma_c \right] + \frac{1}{d} \sqrt{\frac{2(d+1)}{d}} \mathbf{r} \cdot \boldsymbol{\sigma} \quad (\text{by equation (18)}) \\ &= \frac{1}{d^2} + \frac{d+1}{d^2} \mathbf{r}^2 + \frac{d+1}{2d} (\mathbf{r} * \mathbf{r}) \cdot \boldsymbol{\sigma} + \frac{1}{d} \sqrt{\frac{2(d+1)}{d}} \mathbf{r} \cdot \boldsymbol{\sigma} \\ &= \frac{1}{d} \left(\frac{1}{d} + \frac{d+1}{d} \mathbf{r}^2 \right) + \sqrt{\frac{d+1}{2d}} \left(\sqrt{\frac{d+1}{2d}} (\mathbf{r} * \mathbf{r}) + \frac{2}{d} \mathbf{r} \right) \cdot \boldsymbol{\sigma} \end{aligned}$$

Now we impose that this linear combination of I_d , the identity matrix of order d , and the σ_a matrices is equal to the linear combination of these same matrices in (16). Because the σ_a matrices are traceless and they satisfy (19), we see that the matrices of the set $\{I_d, \sigma_1, \sigma_2, \dots, \sigma_{d^2-1}\}$ are orthogonal, with respect to the Hilbert-Schmidt inner product, so they are linearly independent, in fact, they form a basis of the hermitian matrices of order d , then two linear combinations of these matrices are equal if and only if its coefficients are equal. In our case this means

that

$$\begin{aligned} \frac{1}{d} + \frac{d+1}{d} \mathbf{r}^2 &= 1 & \text{and} \\ \sqrt{\frac{d+1}{2d}} (\mathbf{r} * \mathbf{r}) + \frac{2}{d} \mathbf{r} &= \mathbf{r} \end{aligned}$$

From which we obtain (21) and (22). \square

Equation (16) defines a bijective map

$$\begin{aligned} q_{\boldsymbol{\sigma}} : \mathcal{B}_{d^2-1} &\rightarrow \mathcal{D}_d \\ \mathbf{r} &\mapsto \rho = \frac{1}{d} + \sqrt{\frac{d+1}{2d}} \mathbf{r} \cdot \boldsymbol{\sigma}, \end{aligned} \quad (23)$$

We write $q_{\boldsymbol{\sigma}}$ because this map is fixed once we have chosen the basis for $\mathfrak{su}(d)$. It is interesting to find its inverse

Proposition 7. *The inverse of the map $q_{\boldsymbol{\sigma}}$ is*

$$\begin{aligned} q_{\boldsymbol{\sigma}}^{-1} : \mathcal{D}_d &\rightarrow \mathcal{B}_{d^2-1} \\ \rho &\mapsto \mathbf{r} = \sqrt{\frac{d}{2(d+1)}} \text{Tr}(\rho \boldsymbol{\sigma}). \end{aligned} \quad (24)$$

Proof.

$$\begin{aligned} \rho &= \frac{1}{d} + \sqrt{\frac{d+1}{2d}} r_a \sigma_a \\ \text{Tr}(\rho \sigma_b) &= \text{Tr} \left(\frac{1}{d} \sigma_b + \sqrt{\frac{d+1}{2d}} r_a \sigma_a \sigma_b \right) \\ \text{Tr}(\rho \sigma_b) &= \sqrt{\frac{d+1}{2d}} r_a \text{Tr}(\sigma_a \sigma_b) & \text{(because } \sigma_b \text{ is traceless)} \\ \text{Tr}(\rho \sigma_b) &= \sqrt{\frac{2(d+1)}{d}} r_b & \text{(by (19))} \end{aligned}$$

\square

4. BLOCH REPRESENTATION OF A SIC-POVM

First we define a SIC-POVM, see for example [8]

Definition 2. *A set of positive operators $\{E_i\}_{i=1}^{d^2}$ is a SIC-POVM, for d -dimensional systems, if the following conditions are satisfied*

$$\begin{aligned} E_i &= \frac{1}{d} \rho_i, \quad \text{with } \rho_i \text{ a pure state and } i \in \{1, 2, \dots, d^2\}. \\ \sum_{i=1}^{d^2} E_i &= 1. \\ \text{Tr}(E_j E_i) &= \frac{d\delta_{ij} + 1}{d^2(d+1)}, \quad \text{with } i, j \in \{1, 2, \dots, d^2\}. \end{aligned} \quad (25)$$

From this definition and from the last section we see that the elements of the SIC-POVM can be represented in the following way

$$E_i = \frac{1}{d^2} + \frac{1}{d} \sqrt{\frac{d+1}{2d}} \mathbf{e}_i \cdot \boldsymbol{\sigma}. \quad (26)$$

Where each \mathbf{e}_i satisfies (21) and (22). Our next task is to find what other conditions vectors \mathbf{e}_i satisfy so that the corresponding operators E_i form a SIC-POVM.

Proposition 8. *The vectors $\{\mathbf{e}_i\}_{i=1}^{d^2}$ defined in (26) are the positions of the vertices of a regular and $(d^2 - 1)$ -dimensional simplex.*

Proof.

$$\begin{aligned} \text{Tr}(E_i E_j) &= \text{Tr} \left[\left(\frac{1}{d^2} + \frac{1}{d} \sqrt{\frac{d+1}{2d}} \mathbf{e}_i \cdot \boldsymbol{\sigma} \right) \left(\frac{1}{d^2} + \frac{1}{d} \sqrt{\frac{d+1}{2d}} \mathbf{e}_j \cdot \boldsymbol{\sigma} \right) \right] \\ &= \text{Tr} \left(\frac{1}{d^4} + \frac{d+1}{2d^3} e_{ia} e_{jb} \sigma_a \sigma_b \right) \\ &= \frac{1}{d^3} + \frac{d+1}{d^3} \mathbf{e}_i \cdot \mathbf{e}_j \quad (\text{by (19)}) \\ &= \frac{d\delta_{ij} + 1}{d^2(d+1)} \quad (\text{Imposing the last condition of (25)}). \end{aligned}$$

which implies

$$\mathbf{e}_i \cdot \mathbf{e}_j = \frac{d^2 \delta_{ij} - 1}{(d+1)^2},$$

and we see that the vectors $\{\mathbf{e}_i\}_{i=1}^{d^2}$ satisfy **Definition 1** with $n = d^2 - 1$, $a = (d-1)/(d+1)$, as it should be by (21), and $b = -1/(d+1)^2$. \square

5. PROBABILITY DISTRIBUTIONS CORRESPONDING TO QUANTUM STATES

Let

$$\rho = \frac{1}{d} + \sqrt{\frac{d+1}{2d}} \mathbf{r} \cdot \boldsymbol{\sigma} \quad (27)$$

be a quantum state, mixed or pure. We will find what is the probability distribution over the outcomes of a SIC-POVM, and what is the vector in the corresponding simplex of probability, as described in SECTION 2. The probability distribution is

$$\begin{aligned} p_i &= \text{Tr}(E_i \rho) \\ &= \text{Tr} \left[\left(\frac{1}{d^2} + \frac{1}{d} \sqrt{\frac{d+1}{2d}} \mathbf{e}_i \cdot \boldsymbol{\sigma} \right) \left(\frac{1}{d} + \sqrt{\frac{d+1}{2d}} \mathbf{r} \cdot \boldsymbol{\sigma} \right) \right] \\ &= \text{Tr} \left(\frac{1}{d^3} + \frac{d+1}{2d^2} e_{ia} r_b \sigma_a \sigma_b \right) \\ &= \frac{1}{d^2} + \frac{d+1}{d^2} \mathbf{e}_i \cdot \mathbf{r} \end{aligned} \quad (28)$$

Naming by $\mathcal{Q}_{\mathcal{P}}$ the set of all probability distributions over the outcomes of the SIC-POVM $\{E_i\}_{i=1}^{d^2}$ that correspond to a quantum state, then (28) defines the following bijective map

$$\begin{aligned} m_E : \mathcal{B}_{d^2-1} &\rightarrow \mathcal{Q}_{\mathcal{P}} \subset \mathcal{P}_{d^2} \\ \mathbf{r} &\mapsto \left\{ p_i = \frac{1}{d^2} + \frac{d+1}{d^2} \mathbf{e}_i \cdot \mathbf{r} \right\}_{i=1}^{d^2} \end{aligned} \quad (29)$$

To represent this distribution we can choose any regular simplex, Δ_{d^2} , in \mathbb{R}^{d^2-1} , the most natural is the one defined by the vectors

$$\mathbf{t}_i = (d+1)\mathbf{e}_i \quad i \in \{1, 2, \dots, d^2\},$$

because, as we have seen, the vectors \mathbf{e}_i define themselves a simplex, the factor is needed so that $\|\mathbf{t}_i\| = \sqrt{d^2 - 1}$ in accordance with the norm used in SECTION 2 for these vectors. Now we make use of the bijective map $f_{\Delta_{d^2}}$ defined in SECTION 2 to define the set $\mathcal{Q} = f_{\Delta_{d^2}}(\mathcal{Q}_{\mathcal{P}})$, this set contains therefore all elements of Δ_{d^2}

corresponding to quantum states. We then have the bijective map $f_{\Delta_{d^2}}|_{\mathcal{Q}_{\mathcal{P}}}$ that we will denote by $g_{\mathcal{Q}_{\mathcal{P}}}$ and is defined therefore as

$$g_{\mathcal{Q}_{\mathcal{P}}} : \mathcal{Q}_{\mathcal{P}} \rightarrow \mathcal{Q}$$

$$\{p_i\}_{i=1}^{d^2} \mapsto \mathbf{s} = \sum_{i=1}^{d^2} p_i \mathbf{t}_i. \quad (30)$$

Now we can prove the main result of this paper.

Proposition 9. *The set of Bloch vectors, and the set of elements of Δ_{d^2} corresponding to quantum states are the same set. Namely*

$$\mathcal{B}_{d^2-1} = \mathcal{Q}. \quad (31)$$

Proof. The map $(g_{\mathcal{Q}_{\mathcal{P}}} \circ m_E)$ is a bijection because it is a composition of bijections, it goes from $\mathcal{B}_{d^2-1} \subset \mathbb{R}^{d^2-1}$ to $\mathcal{Q} \subset \mathbb{R}^{d^2-1}$. We therefore need to prove that this map is the identity map, that is

$$(g_{\mathcal{Q}_{\mathcal{P}}} \circ m_E)(\mathbf{r}) = \mathbf{r} \quad \forall \mathbf{r} \in \mathcal{B}_{d^2-1},$$

or, equivalently

$$m_E(\mathbf{r}) = g_{\mathcal{Q}_{\mathcal{P}}}^{-1}(\mathbf{r}).$$

Applying (29) and (11) we obtain

$$\left\{ \frac{1}{d^2} + \frac{d+1}{d^2} \mathbf{e}_i \cdot \mathbf{r} \right\}_{i=1}^{d^2} = \left\{ \frac{1}{d^2} (\mathbf{t}_i \cdot \mathbf{r} + 1) \right\}_{i=1}^{d^2}$$

$$\left\{ \frac{1}{d^2} [(d+1) \mathbf{e}_i \cdot \mathbf{r} + 1] \right\}_{i=1}^{d^2} = \left\{ \frac{1}{d^2} (\mathbf{t}_i \cdot \mathbf{r} + 1) \right\}_{i=1}^{d^2},$$

and this last equality is true for all $\mathbf{r} \in \mathcal{B}_{d^2-1}$ because $\mathbf{t}_i = (d+1) \mathbf{e}_i$. \square

Now we can study basic facts about the shape of \mathcal{Q} . We will denote by \mathfrak{P} the subset of \mathcal{Q} corresponding to pure states.

Corollary 3. *\mathfrak{P} is a subset of the $(d^2 - 2)$ -sphere of radius $R_{\mathfrak{P}} = \sqrt{\frac{d-1}{d+1}}$.*

Proof. Because (31) we can use (21) that gives the norm of Bloch vectors corresponding to pure states. \square

Corollary 4. *The sphere that contains \mathfrak{P} is not completely inside Δ_{d^2} , except in the case $d = 2$, then it is the inner sphere of the simplex.*

Proof. We have to prove that the next inequality is true, and that is an equality only if $d = 2$.

$$R_{\mathfrak{P}}^2 \geq R_{in}^2$$

$$\frac{d-1}{d+1} \geq \frac{1}{d^2-1} \quad (\text{by (10)})$$

$$d-1 \geq \frac{1}{d-1} \quad (\text{multiplying by } d+1)$$

$$(d-1)^2 \geq 1 \quad (\text{multiplying by } d-1)$$

\square

The following result was also obtained in [1].

Proposition 10. *The sphere that contains \mathfrak{P} is tangent to the facets of Δ_{d^2} of dimension $m_{\mathfrak{P}} = \frac{(d+2)(d-1)}{2}$.*

Proof. First observe that $m_{\mathfrak{P}}$ is a natural number because $d + 2$ and $d - 1$ have opposite parity, so one of them is divisible by 2. Equation (7) gives the distance from the origin to the m -facets, we will demonstrate that d_m , for the particular value $m = m_{\mathfrak{P}}$, is the radius of the sphere that contains \mathfrak{P} .

$$\begin{aligned} d_{m_{\mathfrak{P}}} &= \sqrt{\frac{d^2 - 1 - m_{\mathfrak{P}}}{m_{\mathfrak{P}} + 1}} \\ &= \sqrt{\frac{d^2 - 1 - \frac{(d+2)(d-1)}{2}}{\frac{(d+2)(d-1)}{2} + 1}} \\ &= \sqrt{\frac{d-1}{d+1}} \end{aligned}$$

□

Therefore \mathcal{Q} is a subset of a $(d^2 - 1)$ -ball truncated by the m -facets of Δ_{d^2-1} with $m > m_{\mathfrak{P}} = \frac{(d+2)(d-1)}{2}$, because from (9) we have that if $m > m_{\mathfrak{P}}$ then $d_m < d_{m_{\mathfrak{P}}}$. But the shape of \mathcal{Q} is not simply this truncated ball, as we have emphasized it is a proper subset of this body. Remember that the pure states are a $(2d - 2)$ -dimensional manifold, so not all points on the surface of the ball, even those situated inside the simplex, can be quantum states.

The following result can be found in [3], although it is obtained in a different way.

Corollary 5. *If $\{p_i\}_{i=1}^{d^2}$ is the distribution of probability over the outcomes of a SIC-POVM of a pure state then*

$$\sum_{i=1}^{d^2} p_i^2 = \frac{2}{d(d+1)}. \quad (32)$$

Proof. We simply use **Corollary 2** and the last proposition.

$$\begin{aligned} \sum_{i=1}^{d^2} p_i^2 &= \frac{1}{m_{\mathfrak{P}} + 1} \\ &= \frac{1}{\frac{(d+2)(d-1)}{2} + 1} \\ &= \frac{2}{d(d+1)}. \end{aligned}$$

□

6. CONCLUSIONS AND FUTURE RESEARCH

We have proved that with suitable scalings the set of Bloch vectors, \mathcal{B} , is equal to the set of points, \mathcal{Q} , of the simplex associated to the probability distributions over the outcomes of a SIC-POVM that correspond to quantum states. We have seen that \mathcal{Q} is a subset of a $d^2 - 1$ -ball truncated by the m -facets of a $d^2 - 1$ -simplex with $m > m_{\mathfrak{P}} = \frac{(d+2)(d-1)}{2}$ (d is the dimension of the Hilbert space we are considering). As a consequence for pure states $\sum_{i=1}^{d^2} p_i^2 = \frac{2}{d(d+1)}$, where p_i is the probability of obtaining result i when measuring the SIC-POVM $\{E_i\}_{i=1}^{d^2}$.

The final objective of this work is to understand in physical, not purely mathematical, terms why \mathcal{Q} has that structure. Why are the pure states situated on a sphere? Why is this sphere tangent to some of the facets of our simplex Δ_{d^2} ? Why are these facets precisely those of dimension $\frac{(d+2)(d-1)}{2}$?

I think that trying to answer these questions we will have a deeper understanding of Quantum Foundations, and therefore of our world.

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